Midterm 1 Review Questions

There will be seven multiple-choice questions and three work-out problems in Midterm 1.

For a review of material about the concepts covered in Midterm 1, please refer to the pdf named "Formula sheet Midterm 1" in the content folder. Below are some selected questions from the final exams from 2018 to 2019. You can find more exercises from the past exam archive.

Fall 2018 #1, #2, #4, #6, #9, #14
Fall 2019 #1, #2, #8, #10, #13, #16
Spring 2018 #2, #4, #12,
Spring 2019 #2, #4

We list the above questions here for convenience. The complete notes for solving these questions will be posted on Thursday, Feb 17.

Fall 2018 Existence and Uniqueness Theorem

#1. Consider the system of linear equations

$$\begin{aligned} x + 2y + 3z &= 1 \\ 3x + 5y + 4z &= a \\ 2x + 3y + a^2 z &= 0. \end{aligned}$$

For which value of a is the system inconsistent?

A. $a = -1$
B. $a = 2$
C. $a = 1$
D. $a = -2$
E. $a = 3$

Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot columnthat is, if and only if an echelon form of the augmented matrix has no row of the form $\begin{bmatrix} 0 & \cdots & 0 & b \end{bmatrix}$ with b nonzero.

If a linear system is consistent, then the solution set contains either
(i) a unique solution, when there are no free variables, or
(ii) infinitely many solutions, when there is at least one free variable.

The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 3 & 5 & 4 & a \\ 2 & 3 & a^2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -5 & a-3 \\ 0 & -1 & a^2-6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -5 & a-3 \\ 0 & 0 & a^2-1 & 1-a \end{bmatrix}$$

The system is inconsistent $\Leftrightarrow$ the last row is of the form [0 0 0 nonzero]

$$\begin{cases} a^2-1=0 \\ 1-a \neq 0 \end{cases} \Rightarrow \begin{cases} a=1 \text{ or } a=-1 \\ a \neq 1 \end{cases} \Rightarrow a=-1$$

Solutions for Ax=b

3 eqns 2 unknowns

#2. Consider the equation $A\mathbf{x} = \mathbf{b}$ where A is a 3×2 matrix and $\mathbf{b}$ is in $\mathbb{R}^3$. Which of the following statements is true for every matrix A?

A. The equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for every $\mathbf{b}$ in $\mathbb{R}^3$.

B. Whenever the equation $A\mathbf{x} = \mathbf{b}$ is consistent, it has exactly one solution $\mathbf{x}$.

C. Whenever the equation $A\mathbf{x} = \mathbf{b}$ is consistent, it has infinitely many solutions $\mathbf{x}$.

D. The equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for at least one $\mathbf{b}$ in $\mathbb{R}^3$.

E. If the columns of A are a scalar multiple of one another, then the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

A. Not true. Eg: A: zero matrix, $\vec{b} = \vec{0}$

B. Not true Eg: $\begin{cases} x_1 + x_2 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$

C. Not true. Eg: $\begin{cases} x_1 = 1 \\ x_2 = 1 \\ 0 = 0 \end{cases}$

D. True. A = zero matrix

$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

E. Not true.
We can take same example in A

The inverse matrix theorem

Linear Independence

#4. Which of the following subsets of $\mathbb{R}^3$ is linearly independent?

A. $\left\{ \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \cancel{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix} \right\}$ since it has a zero vector

B. $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 18 \\ 12 \\ 6 \end{bmatrix} \right\}$ X They are multiples of each other.

C. $\left\{ \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix} \right\}$ X Notice det A = 0, where A is constructed using the vectors (The Inverse Matrix Thm)

D. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ X

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

E. $\left\{ \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} \right\}$

$$\begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix has 3 pivot positions

$\Leftrightarrow$ columns are linearly independent.

Basis for Nul A

#6. Find a basis for the null space of $A = \begin{bmatrix} -3 & 6 & -1 & -4 \\ 1 & -2 & 2 & 3 \\ 2 & -4 & 5 & 7 \end{bmatrix}$.

A. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

B. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

C. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.

D. $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

E. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

Nul A: - The *null space* of a matrix A is the set Nul A of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.
- To test whether a given vector $\mathbf{v}$ is in Nul A, just compute $A\mathbf{v}$ to see whether $A\mathbf{v}$ is the zero vector.
- To find a basis for Nul A, we solve the equation $A\mathbf{x} = \mathbf{0}$ and write the solution for $\mathbf{x}$ in parametric vector form. The vectors in the parametric form give us a basis for Nul A.

We solve $A\vec{x} = \vec{0}$. The augmented matrix

$$\left[\begin{array}{cccc|c} -3 & 6 & -1 & -4 & 0 \\ 1 & -2 & 2 & 3 & 0 \\ 2 & -4 & 5 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc} 1 & -2 & 2 & 3 & 0 \\ -3 & 6 & -1 & 4 & 0 \\ 2 & -4 & 5 & 7 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & 0 \\ 0 & 0 & 5 & 5 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = 2x_2 - x_4 \\ x_2 = x_2 \\ x_3 = -x_4 \\ x_4 = x_4 \end{cases} \Rightarrow \vec{x} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

basic: x_1, x_3. free: x_2 x_4

Compute the inverse

#9. Let

Check Spring 2018 #4 for another method.

$$A = \begin{bmatrix} 3 & 5 & 8 \\ -2 & -2 & -9 \\ 2 & 4 & 5 \end{bmatrix}$$

we have the (i, j)-entry of A^{-1} given by

$$(A^{-1})_{i,j} = \frac{1}{\det(A)} C_{j,i}.$$

Cofactor: $C_{ij} = (-1)^{i+j} \det A_{ij}$

Then the $(1, 2)$-entry of A^{-1} is:

A. 6

B. $-\frac{7}{6}$

C. $\frac{7}{6}$

D. $-\frac{4}{3}$

E. $\frac{4}{3}$

Recall the (1,2)-entry of A^{-1}: $(A^{-1})_{12} = \frac{1}{\det A} C_{21}$

$$\det A = 3 \cdot \begin{vmatrix} -2 & -9 \\ 4 & 5 \end{vmatrix} - 5 \begin{vmatrix} -2 & -9 \\ 2 & 5 \end{vmatrix} + 8 \begin{vmatrix} -2 & -2 \\ 2 & 4 \end{vmatrix}$$

$$= 3 \cdot (-10 + 36) - 5(-10 + 18) + 8 \cdot (-8 + 4)$$

$$= 3 \cdot 26 - 5 \cdot 8 - 8 \cdot 4$$

$$= 78 - 40 - 32$$
$$= 6$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 5 & 8 \\ 4 & 5 \end{vmatrix} = -(25-32) = 7$$

Then the $(1,2)$-entry for A^{-1} is

$$\frac{1}{\det A} C_{21} = \frac{7}{6}$$

Onto and one-to-one transformation

#14. Let T be the linear transformation whose standard matrix is $\begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} = A$ 3×4. Which of the following statements are true?

~~(i) T maps $\mathbb{R}^3$ onto $\mathbb{R}^4$~~

(ii) T maps $\mathbb{R}^4$ onto $\mathbb{R}^3$ ✓

(iii) T is onto ✓

~~(iv) T is one-to-one~~

A. (i) and (iii) only

B. (i) and (iv) only

C. (ii) and (iv) only

D. (ii) and (iii) only

E. (ii), (iii) and (iv) only

Onto and One-to-One Linear Transformations	
Onto:	- A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$. This is an existence question. - Let A be the standard matrix for T, then T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$ (if and only if A has a pivot position in every row).
One-to-One:	- A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$. This is a uniqueness question. - T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution. - Let A be the standard matrix for T, then T is one-to-one if and only if the columns of A are linearly independent.

- The matrix has size 3×4. So $T : \mathbb{R}^4 \to \mathbb{R}^3$

since $$A_{3\times 4}\, \vec{x}_{4\times 1} = \vec{y}_{3\times 1}$$

- A has a pivot position in every row. So the columns of A spans $\mathbb{R}^3$.

- A has 4 columns in $\mathbb{R}^3$. They cannot be linearly independent! So T is not one-to-one.

Fall 2019 **Properties of the determinant**

#1. Given the determinant

$$\begin{vmatrix}2 & d & a+3d\\ 2 & e & b+3e\\ 10 & 5f & 5c+15f\end{vmatrix} = 120$$

what is the determinant

$$\begin{vmatrix}a & b & c\\ d & e & f\\ 1 & 1 & 1\end{vmatrix}?$$

A. -4
B. 12
C. 120
D. -12
E. 4

$$\begin{bmatrix}2 & d & a+3d\\ 2 & e & b+3e\\ 10 & 5f & 5c+15f\end{bmatrix} \xrightarrow{\text{Transpose}} \begin{bmatrix}2 & 2 & 10\\ d & e & 5f\\ a+3d & b+3e & 5c+15f\end{bmatrix}$$

$120 \longrightarrow 120$

$$\xrightarrow{R3\to R3-3R2} \begin{bmatrix}2 & 2 & 10\\ d & e & 5f\\ a & b & 5c\end{bmatrix} \xrightarrow{R1\leftrightarrow R3} \begin{bmatrix}a & b & 5c\\ d & e & 5f\\ 2 & 2 & 10\end{bmatrix}$$

$120 \longrightarrow 120$ $\qquad$ $120 \to -120$

$$\xrightarrow{R3\to \frac{1}{2}R3} \begin{bmatrix}a & b & 5c\\ d & e & 5f\\ 1 & 1 & 5\end{bmatrix} \xrightarrow{Col3\to \frac{1}{5}Col3} \begin{bmatrix}a & b & c\\ d & e & f\\ 1 & 1 & 1\end{bmatrix}$$

$-120 \to \frac{-120}{2} = -60$ $\qquad$ $-60 \to -\frac{60}{5} = -12$

Solutions to Ax=b

#2. Let A be an $m\times n$ matrix and $\mathbf{b}$ be a non-zero vector in $\mathbb{R}^m$. Which of the following statements must be TRUE?

m eqns, n unknowns.

(i) If $A\mathbf{x}=\mathbf{0}$ has only the trivial solution, then $A\mathbf{x}=\mathbf{b}$ has no solution.
(ii) If $A\mathbf{x}=\mathbf{b}$ has exactly one solution, then A is an $m\times n$ matrix with $m\ge n$.
(iii) If $A\mathbf{x}=\mathbf{0}$ has infinitely many solutions, then $A\mathbf{x}=\mathbf{b}$ has infinitely many solutions.
(iv) If A is an $n\times n$ square matrix, then $A\mathbf{x}=\mathbf{0}$ has exactly one solution.

A. (ii) only
B. (iii) only
C. (i) and (ii) only
D. (iii) and (iv) only
E. (ii), (iii), and (iv) only

(i) Not true. $A = I_2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $\vec{b} = \begin{bmatrix}1\\ 0\end{bmatrix}$. Then $A\vec{x}=\vec{0}$ has only trivial solution. But $A\vec{x}=\vec{b} \Rightarrow \begin{cases}x_1=1\\ x_2=0\end{cases}$

(ii). True. Unique solution $\Rightarrow$ no free variables

$\Rightarrow$ A has at least n pivot positions.

Since each pivot position corresponds to one row $\Rightarrow m \geq n$

(iii) Not true. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A\vec{x} = \vec{0}$ has infinitely many solution. but

$A\vec{x} = \vec{b}$. has no solutions.

(iv) Not True. If A is a zero matrix

then $A\vec{x} = \vec{0}$ is always true. We have

infinite'y many solutions in this case.

Basis for Col A and Nul A

#8. Suppose $A = \begin{bmatrix} -1 & -3 & 0 & -4 & -6 \\ -1 & -1 & -1 & -1 & -3 \\ 2 & -1 & 2 & -4 & 0 \end{bmatrix}$. Which of the following statements is false?

A. $\left\{ \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\}$ is a basis for Col A. ✓

B. $\left\{ \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ -4 \end{bmatrix} \right\}$ is a basis for Col A.

C. $\left\{ \begin{bmatrix} 2 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul A. ✓

D. $\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul A. ✗

E. $A\mathbf{x} = \begin{bmatrix} 5 \\ 2 \\ -8 \end{bmatrix}$ is consistent.

Col A, Nul A

Col A: - The *column space* of a matrix A is the set Col A of all linear combinations of the columns of A.
- The pivot columns of a matrix A form a basis for the column space of A.

Nul A: - The *null space* of a matrix A is the set Nul A of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.
- To test whether a given vector $\mathbf{v}$ is in Nul A, just compute $A\mathbf{v}$ to see whether $A\mathbf{v}$ is the zero vector.
- To find a basis for Nul A, we solve the equation $A\mathbf{x} = \mathbf{0}$ and write the solution for $\mathbf{x}$ in parametric vector form. The vectors in the parametric form give us a basis for Nul A.
- The nullity of a matrix A is the dimension of its Nul$A$.

Find basis for Nul A: We solve $A\vec{x} = \vec{0}$.

$$[A \ \vec{0}] \sim \left[\begin{array}{ccccc|c} -1 & -1 & -1 & -1 & -3 & 0 \\ -1 & -3 & 0 & -4 & -6 & 0 \\ 0 & -3 & 0 & -6 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} -1 & -1 & -1 & -1 & -3 & 0 \\ 0 & -2 & 1 & -3 & -3 & 0 \\ 0 & 1 & 0 & 2 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 2 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Compute $A \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \neq \vec{0}$!

So it cannot be in Nul A.

Basic varibles: x_1, x_2, x_3. free varibles are x_4 x_5.

Pivot columns are 1, 2, 3.

$$\begin{cases} x_1 - 2x_4 = 0 \\ x_2 + 2x_4 + 2x_5 = 0 \\ x_3 + x_4 + x_5 = 0 \end{cases}$$

$\Rightarrow$ A basis for Col A is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\}$

$$\Rightarrow \begin{cases} x_1 = 2x_4 \\ x_2 = -2x_4 - 2x_5 \\ x_3 = -x_4 - x_5 \end{cases}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_4 \\ -2x_4 - 2x_5 \\ -x_4 - x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} 2 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Homogeneous equation Ax=0

#13. Consider the equation $A\mathbf{x} = \mathbf{0}$, where A is a 5×7 matrix. Which of the following statements is/are TRUE?

5 eqns, 7 unknowns.

(i) $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

(ii) The matrix A has $\operatorname{rank}(A) \leq 5$.

(iii) The associated linear system has exactly two free variables.

A. (i) and (ii) only

B. (i) and (iii) only

C. (i) only

D. (ii) only

E. (i), (ii), and (iii)

(i) True. The eqn has 7 unknowns. but a most 5 pivot position (corresponds to basis varibles).

(ii) True. Rank A = dim(Col(A)). The basis of A corresponds to the pivot positions of A. which is at most 5.

(iii) Not true. Eg:

$$\begin{cases} x_1 = x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

Linear transformation

#16. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a \\ 1 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ a \\ 4 \end{bmatrix}$$

$\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$

where a is a real number. Which of the following statements are/is TRUE?

(i) The linear transformation T is one-to-one for every real value of a.

(ii) The linear transformation is not onto for $a = 2$.

(iii) The standard matrix for the linear transformation T (relative to the standard basis on $\mathbb{R}^3$) has rank 3 for all real numbers $a \neq 2, 8$.

A. (ii) only

B. (i) and (ii) only

C. (iii) only

D. (ii) and (iii) only

E. (i), (ii) and (iii)

Observation: If we compute det A and use the Invertible Matrix Thm, we can check the statements. (i) (ii) (iii) together. Since we only need to check when det (A) = 0.

The standard matrix for T is:

$$A = \begin{bmatrix} a & -1 & 4 \\ 1 & 3 & a \\ 2 & 1 & 4 \end{bmatrix}$$

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent.

1. A is an invertible matrix.
2. A is row equivalent to the $n \times n$ identity matrix.
3. A has n pivot positions.
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
5. The columns of A form a linearly independent set.
6. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one. (i)
7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
8. The columns of A span $\mathbb{R}^n$.
9. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. (ii)
10. There is an $n \times n$ matrix C such that $CA = I$.
11. There is an $n \times n$ matrix D such that $AD = I$.
12. A^T is an invertible matrix.
13. The columns of A form a basis of $\mathbb{R}^n$.
14. $\operatorname{Col} A = \mathbb{R}^n$.
15. $\operatorname{rank} A = n$. (iii)
16. $\dim \operatorname{Nul} A = 0$.
17. $\operatorname{Nul} A = \{\mathbf{0}\}$.
18. $\det A \neq 0$.

$\det A$

$$= a\begin{vmatrix} 3 & a \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 1 & a \\ 2 & 4 \end{vmatrix} + 4\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= a(12-a) + (4-2a) + 4\cdot(-5)$$

$$= 12a - a^2 + 4 - 2a - 20$$

$$= -a^2 + 10a - 16$$

So $\det(A) = -a^2 + 10a - 16$.

Then $\det(A) = 0 \iff a^2 - 10a + 16 = (a-2)(a-8) = 0$

$\iff a = 2$ or $a = 8$.

So if $a \neq 2, 8$. $\det A \neq 0$.

Spring 2018 The inverse matrix theorem

#2. Let A be an $n \times n$ singular real matrix. Which of the following statements are always true?

det A = 0

(i) $\det(A) = 0$ ✓

(ii) A is row equivalent to the identity matrix ✗

(iii) $A\mathbf{x} = \mathbf{0}$ must have nontrivial solutions ✓

(iv) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$ ✗ Assume $\vec{b} = \vec{0}$

A. (i) only

B. (i) and (iii) only

C. (iii) only

D. (ii) and (iv) only

E. (i), (iii), and (iv)

Compute the inverse

#4. Let $A = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. The $(2,1)$ entry (the entry in the second row and the first column) of A^{-1} is

A. $1/2$

B. 1

C. 0

D. $-1/2$

E. -1

Method 1. compute $\frac{1}{\det A} C_{12}$

Method 2.

$$[A \;\; I] = \left[\begin{array}{ccc|ccc} 0 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R1 \leftrightarrow R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R2 \to (-1)\cdot R1 + R2]{R3 \to \frac{R3}{2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R2 \to R3 + R2]{R1 \to R3 \times (-1) + R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \end{array}\right]$$

$$= \left[I \quad A^{-1} \right]$$

The inverse matrix theorem

#12. Find all real number(s) a such that the following vectors form a basis for R^3 (the real vector space of all $n \times 1$ real vectors):

$$\begin{bmatrix} a \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} a \\ 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{Let } A = \begin{bmatrix} a & a & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}.$$

A. $a \neq 1$
B. $a \neq 2$
C. $a = 1$
D. $a = 2$
E. a can be any real number

By the inverse matrix theorem.

$\det A \neq 0 \iff$ columns of A form a basis for $\mathbb{R}^3$.

$$\det A = a \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - a \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= -2a + a + 1$$

$$= -a + 1$$

Thus $\det A \neq 0 \iff a \neq 1$

Spring 2019 The inverse matrix theorem

#2. Suppose the set $\left\{ \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \\ a \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$ is linearly dependent. Find a.

$\det A = 0$

A. $a = -5$.
B. $a = -2$.
C. $a = 2$.
D. $a = 1$.
E. $a = -3$.

Let $A = \begin{bmatrix} -1 & -2 & -1 \\ -3 & -2 & -2 \\ 1 & a & 1 \end{bmatrix}$

$$\det A = (-1) \cdot \begin{vmatrix} -2 & -2 \\ a & 1 \end{vmatrix} - (-2) \begin{vmatrix} -3 & -2 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} -3 & -2 \\ 1 & a \end{vmatrix}$$

$$= -(-2+2a) + 2(-3+2) - (-3a+2)$$

$$= -2a+2-2+3a-2 = a-2 = 0 \Rightarrow a = 2.$$

Linear transformation

Coordinates of x relative to the basis

#4. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$ and $L\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. Find $L\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$.

Note $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$.

A. $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

B. $\begin{bmatrix} 2 \\ 2 \\ 7 \end{bmatrix}$

C. $\begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$

D. $\begin{bmatrix} 0 \\ 6 \\ -3 \end{bmatrix}$

E. $\begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}$

We want to write $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, i.e.

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Augmented matrix: $\left[\begin{array}{cc|c} 1 & 3 & -1 \\ 2 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & \cancel{-4}\ -1 & \cancel{4}\ 1 \end{array}\right]$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

Thus $\begin{cases} x_1 = 2 \\ x_2 = -1 \end{cases}$

So $\begin{bmatrix}-1\\2\end{bmatrix} = 2\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}3\\2\end{bmatrix}$.

Then

$$L\left(\begin{bmatrix}-1\\2\end{bmatrix}\right) = L\left(2\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}3\\2\end{bmatrix}\right).$$

since L is linear

$$= 2L\left(\begin{bmatrix}1\\2\end{bmatrix}\right) - L\left(\begin{bmatrix}3\\2\end{bmatrix}\right)$$

$$= 2\cdot\begin{bmatrix}0\\1\\4\end{bmatrix} - \begin{bmatrix}-2\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}2\\2\\7\end{bmatrix}.$$